Ch. 2: The Special Theory of Relativity

1. Problems with Newtonian Mechanics

(1) Newtonian mechanics fails to describe properly the motion of objects whose speeds approach that of light.

(2) Newtonian mechanics is a limited theory which places no upper limit on speed, and it is contrary to modern experimental results.

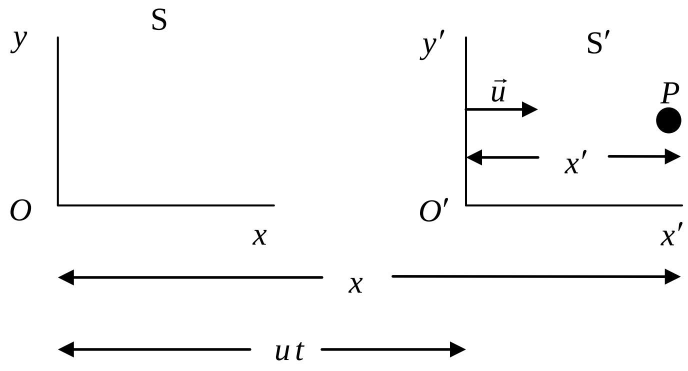
2. Galilean Relativity

An inertial frame of reference is one in which an object subjected to no forces will experience no acceleration. Any system moving at constant velocity with respect to an inertial frame must also be in an inertial frame.

According to the principle of Galilean relativity, the laws of mechanics must be the same in all inertial frames of reference.

An event is some physical phenomenon. To describe a physical event, a frame of reference must be established. There is no absolute inertial frame of reference. This means that the results of an experiment performed in a vehicle moving with uniform velocity will be identical to the results of the same experiment performed in a stationary vehicle.

Consider two inertial frames, S and S', whose origins coincide at $t = 0$. S' moves to the right with constant velocity $\vec{u}$ measured relative to S. An observer O in S describes the event (the moving particle) with space-time coordinates (x, y, z, t), while an observer O' in S' describes the same event with space-time coordinates (x', y', z', t').



The Galilean coordinate transformation equations give the relationship among the coordinates:

$$\boxed{x' = x - ut\,; \quad y' = y\,; \quad z' = z\,; \quad t' = t.}$$

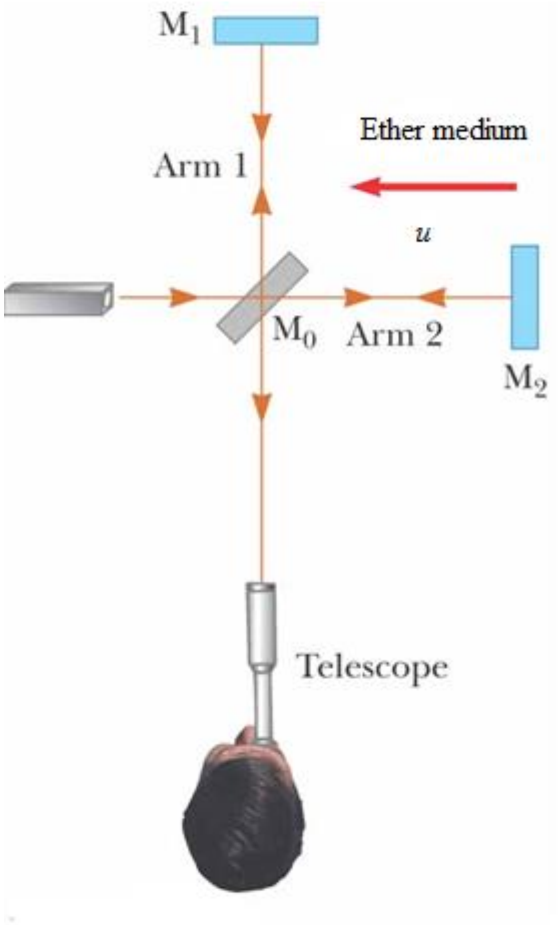
Note that the time at which an event occurs for an observer in S is the same as the time for the same event in S′, and this turns out to be incorrect when *u* is comparable to the speed of light. If the S′ frame is moving in the negative *x* direction relative to S, then we replace *u* by – *u*.

The Galilean velocity transformation equations give the particle velocity as observed by *O* and *O*′:

$$\boxed{v'_x = v_x - u\,; \quad v'_y = v_y\,; \quad v'_z = v_z.}$$

3. The Michelson-Morley Experiment

Maxwell showed the speed of light in free space is $c = 3.00 \times 10^8$ m/s. Physicists in the late 1800s thought light moved through a medium called the ether, and the speed of light would be *c* only in a special absolute frame at rest with respect to the ether.



The Michelson interferometer was used to detect small changes in the speed of light by determining the velocity of the Earth relative to the ether. According to the Galilean velocity transformation equation, the speed of light measured in the Earth frame should be $c-u$ as the light approaches mirror M_2, and $c+u$ as the light is reflected from mirror M_2. Measurements failed to show any change in the fringe pattern and the idea of ether was discarded.

Light is now understood to be an electromagnetic wave, which requires no medium for its propagation.

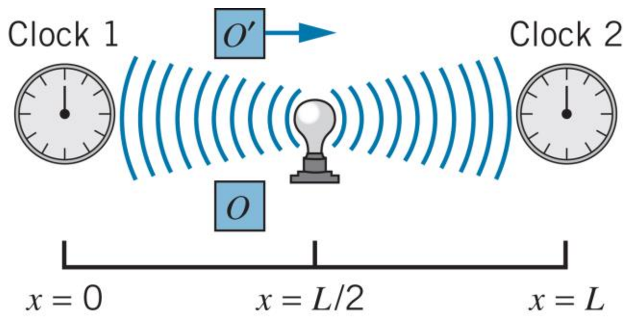
4. Einstein's Postulates

(1) The principle of relativity: the laws of physics must be the same in all inertial reference frames**.**

The first postulate means that the results of any kind of experiment performed in a laboratory at rest must be the same as when performed in a laboratory moving at a constant speed relative to the first one.

(2) The constancy of the speed of light: the speed of light in a vacuum has the same value c in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

5. Simultaneity

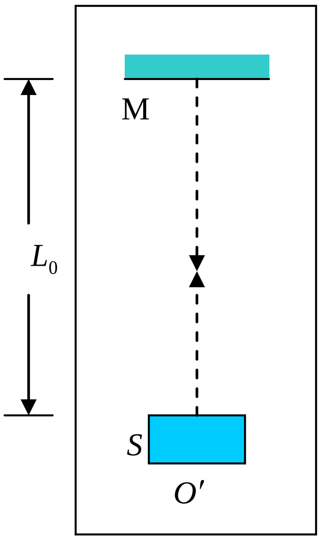


A flash of light, emitted from a lamp midway between the two clocks, starts the two clocks simultaneously according to O. Observer O' sees clock 2 start earlier than clock 1. Therefore, two events that are simultaneous in one reference frame are not simultaneous in a second reference frame moving relative to the first; that is, simultaneity is not an absolute concept, but rather one that depends on the state of motion of the observer.

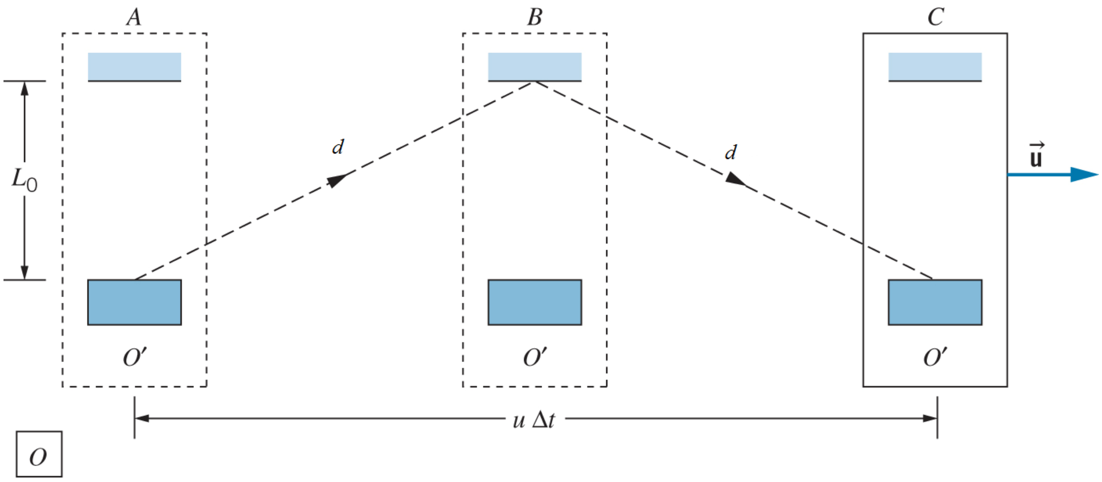
6. Consequences of Einstein's Postulates

• Time Dilation

The source S emits a flash of light directed at the mirror (event 1) and the flash arrives back (event 2) after being reflected by the mirror M. Observer O' is at rest in his frame and carries a clock to measure the time between the events Δt_0 (the proper time). He observes the events to occur at the same location.



$$\Delta t_0 = 2L_0/c.$$



Observer O is a stationary observer on the Earth. He observes the clock carried by O' to move with speed u. By the time the light from the source reaches the mirror, the mirror has moved to the right. Therefore, the light must travel farther with respect to O than with respect to O'. Because O observes the light beam to travel at speed c, the time interval measured by O is

$$\Delta t = \frac{2d}{c} = \frac{2\sqrt{L_0^2 + (u\Delta t/2)^2}}{c} = \frac{2\sqrt{(c\Delta t_0/2)^2 + (u\Delta t/2)^2}}{c} = \frac{\sqrt{(c\Delta t_0)^2 + (u\Delta t)^2}}{c},$$

$$(c\Delta t)^2 = (c\Delta t_0)^2 + (u\Delta t)^2 \quad \Rightarrow \quad \Delta t\sqrt{c^2 - u^2} = c\Delta t_0,$$

$$\boxed{\Delta t = \frac{\Delta t_0}{\sqrt{1-(u/c)^2}} = \gamma\,\Delta t_0,}$$

where

$$\boxed{\gamma = \frac{1}{\sqrt{1-(u/c)^2}}.}$$

Thus, the time interval Δt between two events measured by an observer moving with respect to a clock is longer than the time interval Δt_0 between the same two events measured by an observer at rest with respect to the clock. This effect is known as time dilation. If a clock is moving with respect to you, the time interval between ticks of the moving clock is observed to be longer than the time interval between ticks of an identical clock in your reference frame.

Note that if $u/c << 1$, then $\gamma \cong 1$; therefore, time dilation is not observed in our everyday lives.

Time dilation is a very real phenomenon that has been verified by various experiments such as airplane flights and muon decay.

• **Length Contraction**

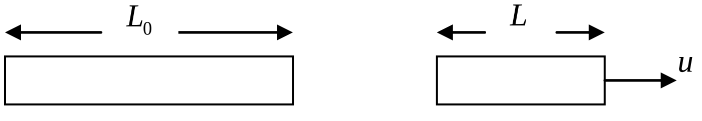
The measured distance between two points depends on the frame of reference of the observer. The proper length L_0 of an object is the length of the object measured by an observer at rest relative to the object. The length of an object measured in a reference frame that is moving with respect to the object is always less than the proper length. This effect is known as length contraction.

Consider a spacecraft traveling with a speed u from one star to another. An observer O at rest on the Earth measures the distance between the stars to be the proper length L_0. According to this observer, the time interval required for the spacecraft to complete the voyage is $\Delta t = L_0 / u$. According to the space traveler O', the time interval to reach the second star is the proper time interval Δt_0, and the distance between the two stars is

$$L = u\,\Delta t_0 = u\,\frac{\Delta t}{\gamma} = \frac{L_0}{\gamma}$$

$$\boxed{L = L_0\sqrt{1-(u/c)^2}\,\cdot}$$

Length contraction takes place only along the direction of motion, and in general, the proper time interval and the proper length are not measured by the same observer.



Example: How fast must an object move before its length appears to be contracted to one-half its proper length?

$$L = \frac{L_0}{2} = L_0 \sqrt{1-(u/c)^2} \ ,$$

$$\frac{1}{4} = 1 - \left(\frac{u}{c}\right)^2 \quad \Rightarrow \quad u = \sqrt{0.75}\, c = 0.866c \, .$$

Example: An astronaut must travel to a distant planet, which is 200 light-years from the Earth. What speed will be necessary if the astronaut wishes to age only 10 years during the round trip?

According to an observer on the Earth, the round-trip travel time $\Delta t \cong 400$ years.

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-(u/c)^2}}$$

$$400 \text{ years} = \frac{10 \text{ years}}{\sqrt{1-(u/c)^2}} \ ,$$

$$1 - \left(\frac{u}{c}\right)^2 = \left(\frac{1}{40}\right)^2 = 6.25 \times 10^{-4} \quad \Rightarrow \quad u = 0.9997c \, .$$

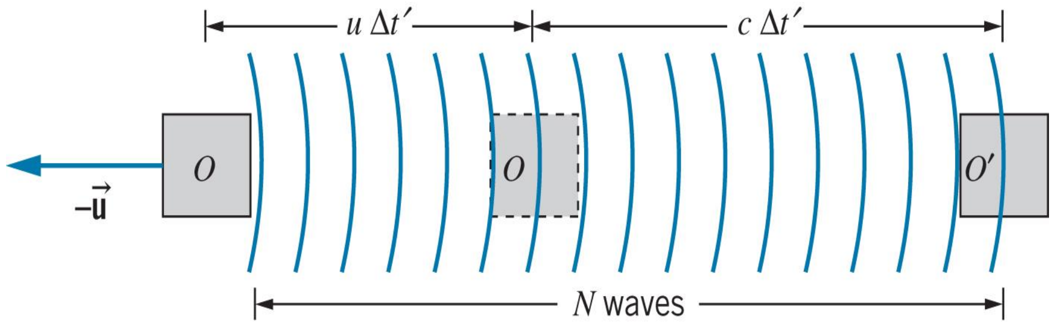
Example: The proper lifetime of a certain particle is 100 ns. (a) How long does it live in the laboratory if it moves at $u = 0.960c$? (b) How far does it travel in the laboratory during that time? (c) What is the distance traveled in the laboratory according to an observer moving with the particle?

(a) $$\Delta t = \frac{\Delta t_0}{\sqrt{1-(u/c)^2}} = \frac{100 \text{ ns}}{\sqrt{1-(0.960)^2}} = 357 \text{ ns}.$$

(b) $$L_0 = u\,\Delta t = (0.960)\,(3\times 10^8 \text{ m/s})\,(357\times 10^{-9} \text{ s}) = 103 \text{ m}.$$

(c) $$L = u\,\Delta t_0 = (0.960)\,(3\times 10^8 \text{ m/s})\,(100\times 10^{-9} \text{ s}) = 28.8 \text{ m}.$$

• **The Relativistic Doppler Effect**



A source of electromagnetic waves, at rest in the reference frame of *O*, moves at a speed *u* away from observer O'. Suppose *O* observes the source to emit *N* waves at frequency *f*. According to *O*, it takes an interval $\Delta t_0 = N/f$ for these *N* waves to be emitted. The corresponding time interval to O' is $\Delta t'$, during which O moves a distance $u\,\Delta t'$. The wavelength according to O' is

$$\lambda' = \frac{c\,\Delta t' + u\,\Delta t'}{N} = \frac{c\,\Delta t' + u\,\Delta t'}{f\,\Delta t_0}\,.$$

The frequency according to O' is $f' = c/\lambda'$, so

$$f' = f\,\frac{\Delta t_0}{\Delta t'}\,\frac{1}{1+u/c} = f\,\frac{1}{\gamma\,(1+u/c)} = f\,\frac{\sqrt{1-(u/c)^2}}{1+u/c}$$

$$\boxed{f' = f\,\sqrt{\frac{1-u/c}{1+u/c}}\,.}$$

If the source and observer are approaching one another, replace u by $-u$.

Example: Spectroscopic measurements of light at $\lambda = 397\ \text{nm}$ coming from a distant galaxy reveal a <u>redshift</u> of $\Delta\lambda = 20\ \text{nm}$. What is the recessional speed of the galaxy?

$$\frac{c}{\lambda'} = \frac{c}{\lambda}\sqrt{\frac{1-u/c}{1+u/c}}$$

$$\frac{1}{417} = \frac{1}{397}\sqrt{\frac{1-u/c}{1+u/c}}\,,$$

$$\left(\frac{397}{417}\right)^2 = \frac{1-u/c}{1+u/c} \quad \Rightarrow \quad u = 0.05\,c.$$

7. The Lorentz Transformation

• Coordinate Transformation

These transformation equations are consistent with special relativity and relate the space-time coordinates of *O* to those of O' (moving away form *O* with speed *u* in the xx' direction):

$$\boxed{x' = \frac{x-ut}{\sqrt{1-(u/c)^2}}\,; \quad y' = y\,; \quad z' = z\,; \quad t' = \frac{t-(u/c^2)\,x}{\sqrt{1-(u/c)^2}}\cdot}$$

If O' moves toward *O*, replace *u* with − *u* in the equations. When $u << c$, the Lorentz transformation equations reduce to the Galilean equations.

The Lorentz transformation equations giving the coordinate difference and the time difference between two events are

$$\boxed{\Delta x' = \frac{\Delta x - u\,\Delta t}{\sqrt{1-(u/c)^2}}\,; \quad \Delta t' = \frac{\Delta t-(u/c^2)\,\Delta x}{\sqrt{1-(u/c)^2}}\cdot}$$

Example: Suppose that observer O' carries a clock that he uses to measure a time difference $\Delta t' = \Delta t_0$ between two events occurring at the same place in his reference frame ($\Delta x' = 0$), then

$$\Delta x = u\,\Delta t\,;$$

$$\Delta t' = \Delta t_0 = \frac{\Delta t - (u/c^2)u\,\Delta t}{\sqrt{1-(u/c)^2}} = \Delta t\,\sqrt{1-(u/c)^2} \quad \Rightarrow \quad \Delta t = \frac{\Delta t_0}{\sqrt{1-(u/c)^2}}\,.$$

Example: According to observer *O*, a blue flash occurs at $x_{\mathrm{b}} = 10.4\ \mathrm{m}$ when $t_{\mathrm{b}} = 0.124\ \mu\mathrm{s}$, and a red flash occurs at $x_{\mathrm{r}} = 23.6\ \mathrm{m}$ when $t_{\mathrm{r}} = 0.138\ \mu\mathrm{s}$. According to observer O', who is in motion relative to *O* at speed *u*, the two flashes appear to be simultaneous. Find the speed *u*.

$$t'_{\mathrm{r}} - t'_{\mathrm{b}} = \frac{(t_{\mathrm{r}} - t_{\mathrm{b}}) - (u/c^2)(x_{\mathrm{r}} - x_{\mathrm{b}})}{\sqrt{1-(u/c)^2}}$$

$$0 = 1.4\times10^{-8}\ \mathrm{s} - (u/c^2)\,(13.2\ \mathrm{m}) \quad \Rightarrow \quad u = c\,\frac{(1.4\times10^{-8}\ \mathrm{s})\,(3\times10^{8}\ \mathrm{m/s})}{13.2\ \mathrm{m}} = 0.32c.$$

• Velocity Transformation

The Lorentz velocity transformation equations give the particle velocity as observed by *O* and O':

$$\boxed{v'_x = \frac{v_x - u}{1 - v_x u/c^2}\,; \quad v'_y = \frac{v_y\sqrt{1-(u/c)^2}}{1 - v_x u/c^2}\,; \quad v'_z = \frac{v_z\sqrt{1-(u/c)^2}}{1 - v_x u/c^2}\,.}$$

Proof of the first equation:

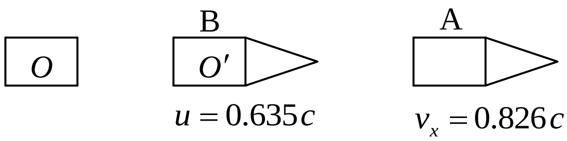
$$dx' = \frac{dx - u\,dt}{\sqrt{1-(u/c)^2}}\,, \quad dt' = \frac{dt - (u/c^2)\,dx}{\sqrt{1-(u/c)^2}}\,;$$

$$v'_x = \frac{dx'}{dt'} = \frac{dx - u\,dt}{dt - (u/c^2)\,dx} = \frac{(dx/dt) - u}{1-(u/c^2)\,(dx/dt)} = \frac{v_x - u}{1 - v_x u/c^2}\,.$$

Special case: If $v_x = c$, then

$$v'_x = \frac{c-u}{1-cu/c^2} = c\,\frac{1-u/c}{1-u/c} = c.$$

Example: Rocket A leaves a space station with a speed of 0.826 *c*. Later, rocket B leaves in the same direction with a speed of 0.635 *c*. What is the velocity of rocket A as observed from rocket B?



$$v'_x = \frac{v_x - u}{1 - v_x u/c^2} = \frac{0.826c - 0.635c}{1-(0.826)(0.635)} = 0.402c.$$

8. Twin Paradox

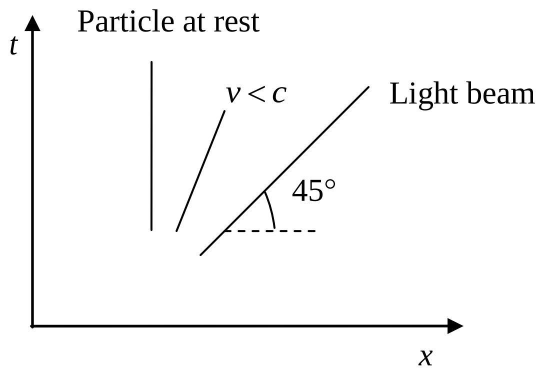
It is a thought experiment involving a set of twins, Casper and Amelia. Amelia travels at 0.6 *c* to Planet X, 6 light years from the Earth. After reaching Planet X, she immediately returns to the Earth at the same speed. When Amelia returns, she has aged 16 years, but Casper has aged 20 years.

Casper's perspective is that he was at rest while Amelia went on the journey. Amelia thinks she was at rest and Casper and the Earth raced away from her and then headed back toward her. The paradox is: which twin has developed signs of excess aging?

The trip in this thought experiment is not symmetrical since Amelia must experience a series of accelerations during the journey. Therefore, Casper can apply the time dilation formula with a time interval of 20 years. This gives a time interval of 16 years for Amelia and this agrees with the earlier result. Therefore, there is no true paradox since Amelia is not in an inertial frame.

9. Space-Time Diagrams

On a space-time diagram, the graph that represents the motion of a particle is called a <u>world-line</u>. The inverse of the slope of the particle's world-line gives its velocity.



10. Relativistic Momentum and Relativistic Kinetic Energy

The relativistic linear momentum of a particle of mass m moving with velocity $\vec{v}$ is

$$\boxed{\vec{p} = \frac{m\vec{v}}{\sqrt{1-(v/c)^2}}}\,.$$

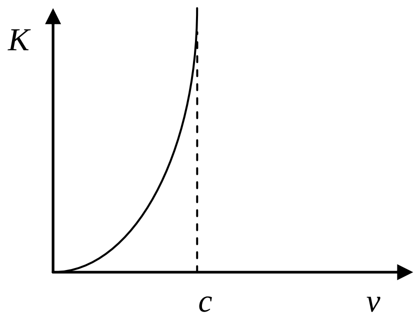
The force acting on the particle is given by

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt}}.$$

The work done by a force on a particle accelerated from rest to a final speed *v* is equal to its kinetic energy:

$$W = K = \int F\,dx = \int dp\,\frac{dx}{dt} = \int v\,dp = v\,p - \int_0^v p\,dv = \frac{mv^2}{\sqrt{1-v^2/c^2}} - \int_0^v \frac{mv\,dv}{\sqrt{1-v^2/c^2}}$$

$$K = \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2} - mc^2$$



$$\boxed{K = \frac{mc^2}{\sqrt{1-(v/c)^2}} - mc^2.}$$

Special case: If $v/c << 1$, then $\left(1-\frac{v^2}{c^2}\right)^{-1/2} \cong 1 + \frac{v^2}{2c^2}$, and we have

$$K \cong mc^2\left(1+\frac{v^2}{2c^2}\right) - mc^2 = \frac{1}{2}mv^2.$$

11. Rest Energy and Relativistic Total Energy

The <u>rest energy</u> E_0 is given by

$$\boxed{E_0 = mc^2.}$$

Example: Find the rest energy of the electron.

$$m_e c^2 = (9.109\times 10^{-31}\ \text{kg})\,(2.998\times 10^{8}\ \text{m/s})^2 = 8.187\times 10^{-14}\ \text{J} = 0.511\,\text{MeV}.$$

Similarly, the rest energy of the proton is 938 MeV.

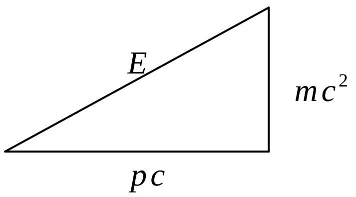
The <u>relativistic total energy</u> *E* of a particle is given by

$$\boxed{E = \frac{mc^2}{\sqrt{1-(v/c)^2}};}$$

therefore,

$$\boxed{E = K + E_0.}$$

We can write



$$\boxed{E = \sqrt{(pc)^2 + (mc^2)^2}.}$$

Proof:

$$(pc)^2 + (mc^2)^2 = \frac{m^2c^2v^2}{1-v^2/c^2} + m^2c^4 = \frac{m^2c^2v^2 + m^2c^4 - m^2c^2v^2}{1-v^2/c^2} = \frac{m^2c^4}{1-v^2/c^2} = E^2.$$

For particles that have zero mass, such as photons, we have

$$\boxed{E = pc\,.}$$

Any particle with energy $E \cong pc$ is called an <u>extreme relativistic</u> particle.

Example: Find the momentum, kinetic energy, and total energy of a proton moving at a speed of 0.756 c.

$$p = \frac{mv}{\sqrt{1-(v/c)^2}} = \frac{mc^2(v/c)}{c\sqrt{1-(v/c)^2}} = \frac{(0.756)(938\,\text{MeV})}{c\sqrt{1-(0.756)^2}} = 1083\,\text{MeV}/c.$$

$$E = \sqrt{(pc)^2 + (mc^2)^2} = \sqrt{(1083\,\text{MeV})^2 + (938\,\text{MeV})^2} = 1433\,\text{MeV}.$$

$$K = E - E_0 = (1433 - 938)\,\text{MeV} = 495\,\text{MeV}.$$

Example: (a) According to observer O, a certain particle has a momentum of 817 MeV/c and a total relativistic energy of 1125 MeV. What is the rest energy of this particle? (b) An observer O' in a different frame of reference measures the momentum of this particle to be 953 MeV/c. What does O' measure for the total relativistic energy of the particle?

(a) $$E_0 = \sqrt{E^2 - (pc)^2} = \sqrt{(1125\,\text{MeV})^2 - (817\,\text{MeV})^2} = 773\,\text{MeV}.$$

(b) $$E' = \sqrt{(p'c)^2 + E_0^2} = \sqrt{(953\,\text{MeV})^2 + (773\,\text{MeV})^2} = 1227\,\text{MeV}.$$

Example: In a nuclear reactor, each atom of uranium-235 (of mass 3.9×10^{-25} kg) releases about 3.2×10^{-11} J when it fissions. What is the change in mass when 1.0 kg of uranium-235 is fissioned?

The number of ^{235}U atoms is $1.0\ \text{kg}/3.9\times10^{-25}\ \text{kg} = 2.6\times10^{24}$. Therefore,

$$\Delta m = \frac{\Delta E_0}{c^2} = \frac{(2.6\times10^{24})\,(3.2\times10^{-11}\ \text{J})}{(3\times10^{8}\ \text{m/s})^2} = 9.2\times10^{-4}\ \text{kg}.$$

12. Conservation Laws in Relativistic Decays and Collisions

In an isolated system of particles, the total linear momentum and the relativistic total energy remain constant.

Example: A π meson of rest energy 139.6 MeV moving at a speed of 0.906 c collides with and sticks to a proton that is at rest. Find the relativistic total energy, the total linear momentum, and the rest energy of the composite particle.

$$E = E_\pi + E_\text{p} = \frac{m_\pi c^2}{\sqrt{1-(v/c)^2}} + m_\text{p} c^2 = \frac{139.6\ \text{MeV}}{\sqrt{1-(0.906)^2}} + 938\ \text{MeV} = 1268\ \text{MeV}.$$

$$p = p_\pi + 0 = \frac{m_\pi v}{\sqrt{1-(v/c)^2}} = \frac{m_\pi c^2 (v/c)}{c\sqrt{1-(v/c)^2}} = \frac{(0.906)(139.6\ \text{MeV})}{c\sqrt{1-(0.906)^2}} = 299\ \text{MeV}/c.$$

$$E_0 = \sqrt{E^2-(pc)^2} = \sqrt{(1268\ \text{MeV})^2-(299\ \text{MeV})^2} = 1232\ \text{MeV}.$$